

Formal similarity between mathematical structures of electrodynamics and quantum mechanics

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Abstract

Electromagnetic phenomena can be described by Maxwell equations written for the vectors of electric and magnetic field. Equivalently, electrodynamics can be reformulated in terms of an electromagnetic vector potential. We demonstrate that the Schrödinger equation admits an analogous treatment. We present a Lagrangian theory of a real scalar field ϕ whose equation of motion turns out to be equivalent to the Schrödinger equation with time independent potential. After introduction the field into the formalism, its mathematical structure becomes analogous to those of electrodynamics. The field ϕ is in the same relation to the real and imaginary part of a wave function as the vector potential is in respect to electric and magnetic fields. Preservation of quantum-mechanics probability is just an energy conservation law of the field ϕ .

1 Introduction

Lagrangian formalism is based on Euler-Lagrange equations of motion. They represent a system of second-order differential equations written for a set of variables that describe the position of a physical system of interest. Hamiltonian formulation suggests an equivalent description in terms of first-order equations written for independent variables describing the position and velocity of the system.

From the beginning, some important equations of physics have been formulated in the Hamiltonian-type form. In particular, a pair

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of Maxwell equations, containing the temporal derivative, represents a Hamiltonian system.

For a mechanical system that obeys the Hamiltonian equations of motion, the existence of a Lagrangian formulation can be proved. For a continuous (field) systems, construction of a Lagrangian formulation is not so trivial procedure, due to presence of spatial derivatives of the fields. For instance, to reformulate the Maxwell equations as the second-order Lagrangian system, one needs to pass from the description in terms of electric \mathbf{E} and magnetic \mathbf{B} fields to those in terms of either the three-dimensional vector potential \mathbf{A} or the four-dimensional vector potential A_μ . The initial fields are related with the three-dimensional potential by differential operators

$$\mathbf{E} = -\frac{1}{c}\partial_t\mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (1)$$

It should be noticed that the importance of the reformulation can not be overestimated, being the starting point for modern formulation of classical and quantum theory of electromagnetic field.

While in some cases it requires the use of rather sophisticated methods, Lagrangian formulations have been found for most fundamental equations of mathematical physics. One of equations which, up to present, is not included into this list, is the Schrödinger equation. In this work we discuss the possibility to construct a Lagrangian formulation for the Schrödinger equation in the sense that we look for the second-order equation for *unique real field* ϕ that would be *equivalent* to the Schrödinger equation¹. After introduction the field into the formalism, its mathematical structure becomes analogous to those of electrodynamics. In particular, as well as \mathbf{A} represents a potential for magnetic and electric fields, the field ϕ turns out to be a potential for the wave function², giving its real and imaginary parts by differentiation

$$\text{Im } \Psi = \hbar\partial_t\phi, \quad \text{Re } \Psi = -\left(\frac{\hbar^2}{2m}\Delta - V\right)\phi. \quad (2)$$

So, the real field ϕ will be called below the wave-function potential.

The work is organized as follows. Formulation of electrodynamics in terms of the three-dimensional potential is usually achieved by

¹In fact, the problem has been raised already by Schrödinger [1]. Eq. (31) below has been tested by Schrödinger as a candidate for the wave function equation and then abandoned.

²We stress that it does not affects the foundations of Quantum Mechanics. We are talking only on a similarity between the mathematical structures of the two theories.

solving the pair of homogeneous equations of the Maxwell system [2, 3]. In Sect. 2 we proceed in a slightly different way, separating a pair of the Hamiltonian-type equations and replacing the other pair by appropriate initial conditions. Hence the Maxwell system is equivalent to the Hamiltonian system which must be solved under these initial conditions. The Hamiltonian equations can be then turned out into the second order equations for the vector potential. As it will be seen, in this setting, similarity of the electrodynamics and the quantum-mechanics formulations became transparent. In Sect. 3 we apply a similar procedure to the Schrödinger equation, obtaining the Lagrangian action for the wave-function potential and demonstrate the equivalence of equation of motion for the potential to the Schrödinger equation. In the Conclusion we compare the results of two sections and observe the remarkable similarities between the formulations. They are summarized in Figure 1.

2 Electrodynamics in terms of three-dimensional vector potential

Moving electric charges can be described using the charge density $\rho(t, x^a)$ and the current density vector $\mathbf{J}(t, x^a) = \rho(t, x^a)\mathbf{v}(t, x^a)$, where \mathbf{v} is the velocity of a charge at t, x^a . They produce the electric $\mathbf{E}(t, x^a)$ and the magnetic $\mathbf{B}(t, x^a)$ fields. According to Maxwell, an electromagnetic field is described by six functions \mathbf{E}, \mathbf{B} subject to eight equations

$$\frac{1}{c}\partial_t\mathbf{E} - \nabla \times \mathbf{B} = -\frac{1}{c}\mathbf{J}, \quad (3)$$

$$\frac{1}{c}\partial_t\mathbf{B} + \nabla \times \mathbf{E} = 0, \quad (4)$$

$$\nabla \cdot \mathbf{E} = \rho, \quad (5)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (6)$$

We use the notation $\nabla = (\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3})$, $\dot{\varphi} \equiv \partial_t \varphi = \frac{\partial \varphi(t, x^i)}{\partial t}$, $\Delta = \frac{\partial^2}{\partial x^{i2}}$. There are six equations of the first order with respect to time, (3), (4). Two more equations (5), (6) do not involve the time derivative

and hence represent the field analogy of kinematic constraints. We first reduce the number of equations from eight to six.

Maxwell equations as the Hamiltonian system. A specific property of the Maxwell system is that the constraint equations can be replaced by properly-chosen initial conditions for the problem. Indeed, consider the following problem

$$\frac{1}{c}\partial_t \mathbf{E} - \nabla \times \mathbf{B} = -\frac{1}{c}\mathbf{J}, \quad (7)$$

$$\frac{1}{c}\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad (8)$$

with the initial conditions

$$[\nabla \cdot \mathbf{E} - \rho]|_{t=0} = 0, \quad \nabla \cdot \mathbf{B}|_{t=0} = 0. \quad (9)$$

This is equivalent to the problem (3)-(6). Any solution to (3)-(6) satisfies the equations (7)-(9). Conversely, let \mathbf{E} , \mathbf{B} be the solution to the problem (7)-(9). Taking the divergence of Eq. (7), we obtain the consequence³ $\partial_t \nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{J} = \partial_t [\nabla \cdot \mathbf{E} - \rho] = 0$. The initial condition (9) then implies $\nabla \cdot \mathbf{E} - \rho = 0$, that is, Eq. (5). In the same way, taking the divergence of Eq. (8) we arrive at Eq. (6).

Considering \mathbf{B} as the conjugate momentum for \mathbf{E} , the equations (7), (8) can be considered as a Hamiltonian equations

$$\dot{q} = \frac{\delta H(q, p)}{\delta p}, \quad \dot{p} = -\frac{\delta H(q, p)}{\delta q}, \quad (10)$$

where the Hamiltonian is

$$H = \int d^3x \left[\frac{c}{2} \mathbf{B} \cdot \nabla \times \mathbf{B} + \frac{c}{2} \mathbf{E} \cdot \nabla \times \mathbf{E} - \mathbf{B} \cdot \mathbf{J} \right]. \quad (11)$$

From an electric and magnetic fields to the vector potential. To restore the Lagrangian formulation starting from a given Hamiltonian equations (10), it is sufficient to resolve the first equation from (10) with respect to p . Substituting the solution $p(q, \dot{q})$ into the second equation, we obtain the Lagrangian equation for the

³Here we have used the continuity equation. Recall that the charge and current densities in Maxwell equations can not be taken as arbitrary, but must obey the continuity equation $\partial_t \rho + \nabla \cdot \mathbf{J} = 0$. This follows from Eq. (3), if we compute the divergence of both sides, and then use the identity $\nabla \cdot \nabla \times \mathbf{B} = 0$ as well as Eq. (5).

position variable q . Substitution of the solution into the expression $p\dot{q} - H(q, p)$ gives the corresponding Lagrangian, see for example [4].

For the present case, one needs to extract \mathbf{B} from Eq. (7). Due to presence of spatial derivatives of \mathbf{B} , this can not be done by algebraic methods. So, to find a Lagrangian description, we need to reformulate the problem. To make transparent the similarity of the electrodynamics and the quantum mechanics formalisms, we do the reformulation following to a slightly different way as compare with the standard textbooks [2, 3].

It is convenient to unify the vectors \mathbf{E} , \mathbf{B} into the complex field

$$\mathbf{W} \equiv \mathbf{B} + i\mathbf{E}. \quad (12)$$

Then Eqs. (7)-(9) can be written in a more compact form

$$\left(\frac{i}{c}\partial_t + \nabla \times\right) \mathbf{W} = \frac{1}{c}\mathbf{J}, \quad [\nabla \cdot \mathbf{W} - i\rho]|_{t=0} = 0. \quad (13)$$

If we look for a solution of the form $\mathbf{W} = (-\frac{i}{c}\partial_t + \nabla \times)\mathbf{A}$, the equations that appear for \mathbf{A} turn out to be real. They read⁴

$$\frac{1}{c^2}\partial_t^2 \mathbf{A} - \Delta \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \frac{1}{c}\mathbf{J}, \quad (14)$$

$$[\partial_t \nabla \cdot \mathbf{A} + c\rho]|_{t=0} = 0, \quad (15)$$

Hence it is consistent to take \mathbf{A} as a real function. Thus, any real solution $\mathbf{A}(t, x)$ of Eq. (14) determines a solution

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\frac{1}{c}\partial_t \mathbf{A}. \quad (16)$$

of the Maxwell equations.

Conversely, any given solution \mathbf{E} , \mathbf{B} of the Maxwell equations can be written in the form (16), where \mathbf{A} is a solution to the problem (14), (15). To see this, we construct the field

$$\mathbf{A}(t, x^i) = -c \int_0^t \mathbf{E}(\tau, x^i) d\tau + \mathbf{K}(x^i), \quad (17)$$

⁴we use the identity $\epsilon_{cab}\epsilon_{cmn} = \delta_{am}\delta_{bn} - \delta_{an}\delta_{bm}$, then $\nabla \times (\nabla \times \mathbf{A}) = -\Delta \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$.

where \mathbf{K} is any solution to the equation

$$\nabla \times \mathbf{K} = \mathbf{B}(0, x^i). \quad (18)$$

The existence of the solution \mathbf{K} of this equation is guaranteed by the equation (9). By direct substitution, we can verify that the field constructed obeys the equations⁵ (16), (14), (15).

Equations of motion (14) for the vector potential follow from the Lagrangian action

$$S = \int dt d^3x \left[\frac{1}{2c^2} \partial_t \mathbf{A} \cdot \partial_t \mathbf{A} - \frac{1}{2} \nabla \times \mathbf{A} \cdot \nabla \times \mathbf{A} + \frac{1}{c} \mathbf{A} \cdot \mathbf{J} \right]. \quad (20)$$

Note that the solutions $\mathbf{A}(t, x^i)$ and $\mathbf{A}(t, x^i) + \nabla \alpha(x^i)$, where $\alpha(x^i)$ is an arbitrary function, determine the same \mathbf{E} , \mathbf{B} . The action reflects this fact, being invariant under these transformation⁶.

In short, Maxwell equations can be reformulated as a Hamiltonian system that admits a Lagrangian formulation in the appropriately chosen variables.

Maxwell equations as a generalized Hamiltonian system.

For the later use, we mention that the free Maxwell equations can be considered as a generalized Hamiltonian system [5, 6]. Indeed, we can rewrite (7), (8) with $\mathbf{J} = 0$ in the form

$$\partial_t \mathbf{E} = \{\mathbf{E}, H'\}', \quad \partial_t \mathbf{B} = \{\mathbf{B}, H'\}', \quad (21)$$

where H' is the generalized Hamiltonian

$$H' = \int d^3x \frac{c}{2} [\mathbf{E}^2 + \mathbf{B}^2], \quad (22)$$

and the non-canonical Poisson bracket is specified by

$$\{E_i(t, x), B_j(t, y)\}' = -\epsilon_{ijk} \partial_k^x \delta^3(x - y). \quad (23)$$

In contrast to H , the generalized Hamiltonian H' does not contain the spatial derivatives of fields.

⁵we point out that the initial condition (15) can be transformed back into the differential equation. It is easy to see that the problem

$$\frac{1}{c^2} \partial_t^2 \mathbf{A} - \Delta \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) - \frac{1}{c} \mathbf{J} = 0, \quad \partial_t \nabla \cdot \mathbf{A} + c\rho = 0, \quad (19)$$

is equivalent to the problem (14), (15).

⁶The transformation is a reminiscence on a gauge invariance of the four-dimensional formulation that survives in the gauge $A_0 = 0$.

3 Real scalar potential of a quantum-mechanics wave function

Schrödinger equation as a Hamiltonian system. Consider the quantum mechanics of a particle subject to the potential $V(t, x^i)$. The Schrödinger equation for the complex wave function $\Psi(t, x^i)$

$$i\hbar\dot{\Psi} = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi, \quad (24)$$

is equivalent to the system of two equations for two real functions (the real and imaginary parts of Ψ , $\Psi = \varphi + ip$). We have

$$\hbar\dot{\varphi} = -\left(\frac{\hbar^2}{2m}\Delta - V\right)p, \quad (25)$$

$$\hbar\dot{p} = \left(\frac{\hbar^2}{2m}\Delta - V\right)\varphi. \quad (26)$$

We can treat $\varphi(t, x^i)$ and $p(t, x^i)$ as coordinate and conjugate momenta of the field φ at the space point x^i . Then the system has the Hamiltonian form $\dot{\varphi} = \{\varphi, H\}$, $\dot{p} = \{p, H\}$, with the Hamiltonian being

$$H = \frac{1}{2\hbar} \int d^3x \left(\frac{\hbar^2}{2m} (\vec{\nabla}\varphi\vec{\nabla}\varphi + \vec{\nabla}p\vec{\nabla}p) + V(\varphi^2 + p^2) \right). \quad (27)$$

Hence the equations (25), (26) arise from the variation problem with the Hamiltonian action being

$$S_H = \int dt d^3x \left[\pi\dot{\varphi} - \frac{1}{2\hbar} \left(\frac{\hbar^2}{2m} (\vec{\nabla}\varphi\vec{\nabla}\varphi + \vec{\nabla}p\vec{\nabla}p) + V(\varphi^2 + p^2) \right) \right]. \quad (28)$$

Disregarding the boundary term, this can also be rewritten in terms of the wave function Ψ and its complex conjugate Ψ^*

$$S_H = \int dt d^3x \left[\frac{i\hbar}{2} (\Psi^*\dot{\Psi} - \dot{\Psi}^*\Psi) - \frac{\hbar^2}{2m} \vec{\nabla}\Psi^*\vec{\nabla}\Psi - V\Psi^*\Psi \right]. \quad (29)$$

From a wave function to the wave-function potential. Due to the Hamiltonian nature of the Schrodinger equation, it is natural to

search for a Lagrangian formulation of the system (25), (26), that is a second-order equation with respect to the time derivative for the real function $\varphi(t, x^i)$. As it has been already mentioned in Sect. 2, we need to solve (25) with respect to p and then to substitute the result either in Eq. (26) or into the Hamiltonian action (28). This leads immediately to the rather formal non-local expression $p = \hbar(-\frac{\hbar^2}{2m}\Delta - V)^{-1}\partial_t\varphi$. Similarly to the case of electrodynamics, the Schrödinger system cannot be obtained starting from a Lagrangian. Nevertheless, for the case of time-independent potential $V(x^i)$, there is a Lagrangian field theory with the property that any solution to the Schrödinger equation can be constructed from a solution to this theory. To find it we look for solutions of the form

$$\Psi = -(\frac{\hbar^2}{2m}\Delta - V)\phi + i\hbar\dot{\phi}, \quad (30)$$

where $\phi(t, x^i)$ is a *real* function. Inserting (30) into (24) we conclude that Ψ will be a solution to the Schrödinger equation if ϕ obeys the equation

$$\hbar^2\ddot{\phi} + (\frac{\hbar^2}{2m}\Delta - V)^2\phi = 0, \quad (31)$$

which follows from the Lagrangian action

$$S = \int dt d^3x \left[\frac{\hbar}{2}\dot{\phi}\dot{\phi} - \frac{1}{2\hbar}(\frac{\hbar^2}{2m}\Delta - V)\phi(\frac{\hbar^2}{2m}\Delta - V)\phi \right]. \quad (32)$$

This can be treated as the classical theory of field ϕ on the given external background $V(x^i)$. The action contains Planck's constant as a parameter. After the rescaling $(t, x^i, \phi) \rightarrow (\hbar t, \hbar x^i, \sqrt{\hbar}\phi)$ it appears in the potential only, $V(\hbar x^i)$, and thus plays the role of a coupling constant of the field ϕ with the background.

Similarly to electrodynamics, different potentials can lead to the same wave function. It follows from the observation that any function $\alpha(x^i)$ which obeys the equation $(\frac{\hbar^2}{2m}\Delta - V)\alpha = 0$ produces a vanishing wave function (30). So, the potentials ϕ and $\phi + \alpha$ produce the same wave function.

In quantum mechanics the quantity $\Psi^*\Psi$ has an interpretation as a probability density, that is the expression $\Psi^*(t, x^i)\Psi(t, x^i)d^3x$ represents the probability of finding a particle in the volume d^3x

around the point x^i at the instant t . According to the formula (30), we write

$$\Psi^*\Psi = \hbar^2(\dot{\phi})^2 + [(-\frac{\hbar^2}{2m}\Delta + V)\phi]^2 = 2\hbar E, \quad (33)$$

where $E=T + U$ is the energy density of the field φ . Eq. (33) states that the probability density is the energy density of the wave potential ϕ . So the preservation of probability is just an energy conservation law of the theory (32).

Equivalence of equation for the wave-function potential to the Schrödinger equation. Any solution to the field theory (32) determines a solution to the Schrödinger equation according to Eq. (30). We should ask whether an arbitrary solution to the Schrödinger equation can be presented in the form (30). An affirmative answer can be obtained as follows.

Let $\Psi=\varphi+ip$ be a solution to the Schrödinger equation. Consider the expression (30) as an equation for determining ϕ

$$\dot{\phi} = \frac{1}{\hbar}p, \quad (34)$$

$$(\frac{\hbar^2}{2m}\Delta - V)\phi = -\varphi, \quad (35)$$

Here the right-hand sides are known functions. Take Eq. (35) at $t=0$, $(\frac{\hbar^2}{2m}\Delta - V)\phi=-\varphi(0, x^i)$. The elliptic equation can be solved (at least for the analytic function $\varphi(x^i)$ [7]); let us denote the solution as $C(x^i)$. Then the function

$$\phi(t, x^i) = \frac{1}{\hbar} \int_0^t d\tau p(\tau, x^i) + C(x^i), \quad (36)$$

obeys the equations (34), (35). They imply the desired result: any solution to the Schrödinger equation can be presented through the field ϕ and its momenta according to (30). Finally, note that Eqs. (34), (35) together with Eqs. (25), (26) imply that ϕ obeys Eq. (31).

Schrödinger equation as the generalized Hamiltonian system. Let us finish this subsection with one more comment. As we have seen, treating a Schrödinger system as a Hamiltonian one, it

is impossible to construct the corresponding Lagrangian formulation owing to the presence of the spatial derivatives of momentum in the Hamiltonian. To avoid this problem, we can try to treat the Schrödinger system as a generalized Hamiltonian system. We rewrite (25) in the form

$$\dot{\varphi} = \{\varphi, H'\}', \quad \dot{p} = \{p, H'\}', \quad (37)$$

where H' is the generalized Hamiltonian

$$H' = \int d^3x \frac{1}{2\hbar} (p^2 + \varphi^2) = \int d^3x \frac{1}{2\hbar} \Psi^* \Psi, \quad (38)$$

and the non-canonical Poisson bracket is specified by

$$\begin{aligned} \{\varphi, \varphi\}' &= \{p, p\}' = 0, \\ \{\varphi(t, x), p(t, y)\}' &= -\left(\frac{\hbar^2}{2m} \Delta - V\right) \delta^3(x - y). \end{aligned} \quad (39)$$

In contrast to H , the Hamiltonian H' does not contain the spatial derivatives of momentum.

We point out that a non-canonical bracket turns out to be a characteristic property of a singular Lagrangian theory. This observation has been explored in the recent work [8], where we show that there is a singular Lagrangian theory subject to second-class constraints underlying both the Schrödinger equation and the theory of the field ϕ .

4 Conclusion

Results of the present work can be resumed as follows.

1. It has been demonstrated that the Schrödinger equation (24) with time-independent potential $V(x^i)$ is equivalent to the second order equation (31) for the real function ϕ . So, in applications one can look for solutions of the equation (31) instead the Schrödinger one.

The formula (30) implies that after introduction of the field ϕ into the formalism, its mathematical structure becomes analogous to that of electrodynamics. The dynamics of the magnetic \mathbf{B} and electric \mathbf{E} fields is governed by first-order Maxwell equations with respect to the time variable. Equivalently, we can use the vector

Electrodynamics	Quantum mechanics
Maxwell equations can be considered as a Hamiltonian system with the Hamiltonian $\frac{c}{2}(\vec{B}, [\vec{\nabla}, \vec{B}]) + \frac{c}{2}(\vec{E}, [\vec{\nabla}, \vec{E}]) - (\vec{B}, \vec{J})$	Schrödinger equations can be considered as a Hamiltonian system with the Hamiltonian $\frac{\hbar}{4m}(\vec{\nabla}\varphi\vec{\nabla}\varphi + \frac{1}{2\hbar}\vec{\nabla}p\vec{\nabla}p) + V(\varphi^2 + p^2)$
There is a Lagrangian formulation in terms of \vec{A}	There is a Lagrangian formulation in terms of ϕ
\vec{A} represents the potential for magnetic and electric fields, one has $\vec{B} + i\vec{E} = \nabla \times \vec{A} - \frac{i}{c}\partial_t\vec{A}$	ϕ represents the wave-function potential, one has $\Psi = \varphi + ip = -(\Delta - V)\phi + i\hbar\partial_t\phi$
While the Maxwell equations are written in terms of \vec{B} , \vec{E} , the field \vec{E} is the conjugate momenta for \vec{A} but not for \vec{B}	While the Schrödinger equation is written in terms of φ , p , the field p is the conjugate momenta for ϕ but not for φ
The vector potentials \vec{A} and $\vec{A} + \vec{\nabla}\alpha(x^a)$ produce the same electromagnetic field	Scalar potentials ϕ and $\phi + \alpha(x^a)$, where $(\frac{\hbar^2}{2m}\Delta - V)\alpha = 0$ produce the same wave function
Free Maxwell equations form the generalized Hamiltonian system with the Hamiltonian $\frac{c}{2}(\vec{E}^2 + \vec{B}^2)$	Schrödinger equation forms the generalized Hamiltonian system with the Hamiltonian $\frac{1}{2\hbar}(p^2 + \varphi^2)$

Figure 1: Real field ϕ as a wave-function potential.

potential \mathbf{A} , which obeys the second-order equations following from the Lagrangian (20). \mathbf{A} represents the potential for magnetic and electric fields, generating them according to $\mathbf{B} = \nabla \times \mathbf{A}$, $\mathbf{E} = -\frac{1}{c}\partial_t\mathbf{A}$. Similarly to this, the field ϕ turns out to be a potential for the wave function, generating its real and imaginary parts according to $\text{Re } \Psi = -(\frac{\hbar^2}{2m}\Delta - V)\phi$, $\text{Im } \Psi = \hbar\partial_t\phi$.

2. Comparing the reasonings in sections 2 and 3 we observe the remarkable similarities between mathematical structure of electrodynamics and quantum-mechanics. They are summarized in Figure 1 on the page 11.

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